Problem Tutorial: "Hamiltonian "

For $K = 1$, it's just a line, for $K = 2$ it's just a graph on 4 nodes with edges $(1, 2), (2, 3), (3, 1), (3, 4)$.

For $3 \le K \le 20$ we can take a cycle of length K.

Now, consider a clique with n vertices where $n \ge 3$, select some nodes A and B there. Also, consider some chain of length $m \ge 2$, with ends C and D, and connect A to D and B to C.

This graph has exactly $n(n-1)/2-1+(m-1)+2(n-1)$ pairs of nodes between which there is Hamiltonian path. Those are:

- All pairs from the clique except pair (A, B): $\frac{n(n-1)}{2} - 1$ pairs
- Every 2 consecutive nodes in a chain: $2(n - 1)$ pairs
- All pairs (C, X) for X from clique except $X = A$ and all pairs (D, X) for X from clique except $X = B$: $2(m - 1)$ pairs

Luckily, all numbers from 21 to 60 can be presented as $n(n - 1)/2 - 1 + (m - 1) + 2(n - 1)$ for some $n \ge 3, m \ge 2, n + m \le 20$.