# Problem Tutorial: "Two Trees"

Build the centroid decompositions of both trees and denote them as $C_1$ and $C_2$. Consider some pair of vertices $(v, u)$. Let $w_1$ and $w_2$ be the LCAs of $u$ and $v$ in $C_1$ and $C_2$, respectively. Let

$$f(v, w_1, w_2) = d(v, w_1, T_1) + d(v, w_2, T_2).$$

Note that

$$(d(v, u, T_1) + d(v, u, T_2))^2 = (f(v, w_1, w_2) + f(u, w_1, w_2))^2.$$

To calculate the answer for a fixed pair $(w_1, w_2)$, store the following data in each vertex:

- $cnt[v]$ — the count of $v$'s
- $sum_f[v]$ — the sum of $f(v, w_1, w_2)$
- $sum_{f^2}[v]$ — the sum of $f(v, w_1, w_2)^2$

The answer computation can be reduced to some queries of two types: add a vertex — update the above data; get some vertice's contribution to the answer — add

$$cnt[v] \cdot f(v, w_1, w_2)^2 + 2 \cdot sum_f[v] \cdot f(v, w_1, w_2) + sum_{f^2}[v]$$

to the answer. As for the order in which to process the vertices: fix $w_1$ and iterate over only those $v$'s that lie in $w_1$'s subtree in $C_1$. The time complexity of this solution is $O(n \log^2 n)$. Here is a link to a correct code for better understanding: `https://ideone.com/RbNhuw`.

# Problem Tutorial: "Tarzan Jumps"

Let $ans_k$ be the answer for $k$. If we set $H_1$ and $H_N$ to 0, then Tarzan will be able to reach the last tree in 1 jump. Thus, $ans_k \le 2$ for any $k$. So, $ans_k = 0, 1$ or 2. Since $ans_k \ge ans_{k-1}$, then the array $ans$ will look like this: $2, 2, \ldots, 2, 1, 1, \ldots, 1, 0, 0, \ldots, 0$. It follows that it is enough to find such minimal $p_0$ and $p_1$ that $ans_{p_0} = 0$ and $ans_{p_1} = 1$.

1) How to find $p_0$.

Consider a jump from tree $x$ to tree $y$ in which all the intermediate trees are lower than both of those trees (the other case is similar). There are two possibilities:

- $h_x \le h_y$. Then $y$ is the first tree to the right of $x$, which is not lower than $x$.
- $h_x > h_y$. Then $x$ is the first tree to the left of $y$, which is not lower than $y$.

Hence, the number of pairs $(x, y)$ such that the jump from tree $x$ to tree $y$ is possible is of magnitude $O(n)$. For each tree, we can find the closest tree to the right (and left) of it, which is not lower (higher) than it (using a stack). Using this information, we can compute $A_i$ — the minimum number of jumps required to reach the $i$-th tree. Clearly, $p_0 = A_n$.

2) How to find $p_1$.

Let $B_i$ be the minimum number of jumps to reach tree $N$ starting from tree $i$ (it can be computed in the same way as $A_i$). Let $pos$ be the tree the height of which we change.

There are two cases:

1. The shortest path from 1 to $N$ does not visit $pos$. The jump from $i$ to $j$ after one change is possible if initially one of the two conditions was held:
   (a) There was at most one tree between $i$ and $j$ with height at least $\min(h_i, h_j)$.

(b) There was at most one tree between $i$ and $j$ with height at most $\max(h_i, h_j)$.

Let's consider the first case (the second case is similar, you can just multiply all the numbers by $(-1)$).

Let $l_i$ be the second tree to the left of $i$ that is not lower than tree $i$ (if there is no such tree, then let $l_i = 1$). Let $r_i$ be the second tree to the right of $i$ that is not lower than tree $i$ (if there is no such tree, then let $r_i = n$). Tarzan can jump from $i$ to $j$ after one change if and only if $l_j \le i$ and $j \le r_i$ (we need to change the highest tree between $i$ and $j$ for enabling this jump). Among such $i, j$ we need to find those for which $A_i + 1 + B_j$ is minimal. It can be done with a standard scanline algorithm and a segment tree. Initially, in leaf $i$ we store the value $A_i$. After that, we iterate from $j = 1$ to $j = n$. We update $p_1$ with $getmin(l[j], j) + B_j + 1$. And finally, for every $i$ such that $r_i = j$, in leaf $i$ we change the value to $INF$.

2. The shortest path from 1 to $N$ visits $pos$. Let $i$ and $j$ be such that in the shortest path Tarzan jumps from $i$ to $pos$, and then from $pos$ to $j$.

   There are 4 cases:

   (a) All trees between $i$ and $pos$ are lower than $i$ and $pos$, and all trees between $pos$ and $j$ are lower than $pos$ and $j$. We can safely set $H_{pos} = INF$. It is possible to jump from $i$ to $pos$ if and only if $i$ appeared in the stack from part 1 at the moment before we processed tree $pos$ (the stack which was used to find the closest from the left tree that is not lower than the current tree). To find an optimal $i$, we can maintain all values $A_x$ in a set, where $x$ is a tree on the stack. This way, we can find $i$ with minimal $A_i$, from which Tarzan can jump to $pos$. Similarly, find $j$ with minimal $B_j$ and update $p_1$ with $A_i + B_j + (pos \neq 1) + (pos \neq n)$.

   (b) All trees between $i$ and $pos$ are higher than $i$ and $pos$, and all trees between $pos$ and $j$ are higher than $pos$ and $j$. It is similar to case (a).

   (c) All trees between $i$ and $pos$ are lower than $i$ and $pos$, and all trees between $pos$ and $j$ are higher than $pos$ and $j$. For this to happen the following condition must hold:

   $$\max(H_{i+1}, H_{i+2}, .., H_{pos-1}) + 1 < \min(H_{pos+1}, .., H_{j-1}).$$

   Since

   $$\max(H_{i+1}, H_{i+2}, .., H_{pos-1}) < H_i,$$

   then the total number of possible values of max for all $i$'s will be of magnitude $O(n)$. Let's iterate over all pairs $(i, k)$ (such that $H_k = \max(H_{i+1}, H_{i+2}, .., H_{pos-1})$). It is optimal to choose $pos$ as the first tree to the right of $k$ that is not lower than $H_k$. Set $H_{pos} = H_k + 1$. Let $pos_2$ be the first tree to the right of $pos$ that is not higher than $H_k + 1$. Since

   $$\max(H_{i+1}, H_{i+2}, .., H_{pos-1}) + 1 < min(H_{pos+1}, .., H_{j-1}),$$

   then $j \le pos_2$. $j$ must satisfy the following conditions:

   $$pos < j \le pos_2, L_j \le pos,$$

   where $L_j$ is the first tree to the left of $j$ that is not higher than $H_j$. Among all such $j$'s we need to find one with minimal $B_j$, and update $p_1$ with $A_i + B_j + 2$. This can be done offline with a segment tree.

   (d) All trees between $i$ and $pos$ are higher than $i$ and $pos$, and all trees between $pos$ and $j$ are lower than $pos$ and $j$. It is similar to case (c).

# Problem Tutorial: "Inversions"

Solution follows by dynamic programming + Fast Fourier transform in $O(k^2 \log(k))$.

First, understand value $inv(\pi)^k$ as $k$ pairs of inversions of permutation $\pi$, namely $(a_1, b_1), \ldots, (a_k, b_k)$ s.t. $\pi(a_i) > \pi(b_i)$. These pairs may occupy not more that $2k$ positions. We want to calculate the number of pairs $(\pi, M)$, where $M = \{(a_1, b_1), \ldots, (a_k, b_k)\}$ the list of pairs $(a_i, b_i)$ not necessarily distinct. Let us fix the set of distinct indices $S = \{a_1, b_1, \ldots, a_k, b_k\}$ occupied by $M$. Knowing $S$ and $\pi(S)$ (what is on positions from $S$) we can calculate how many completions to full $\pi$ there are, but we need to calc large factorials within it so we should do precalc for every $10^6$ factorial up until the modulo.

So we start with simple dp. Let $a_{i,j}$ denotes number of pairs $(\pi, M)$ where $pi$ is permutation of size $i$ and $M$ is $j$ selected **distinct** (as pairs) inversions. From the state $(i, j)$ we can add new element $i+1$ with $q$ new inversions selected in $\binom{i+1}{q+1}$ ways. Indeed, select $q+1$ elements from $i+1$ - this corresponds to inserting element $i+1$ at leftmost position. Then it could be calculated as

$$a_{i+1,j+q} += a_{i,j} \cdot \binom{i+1}{q+1}$$

in $O(k^3)$ time. But we can do better with FFT in $O(k^2 \log(k))$. Let $a_i(x) = \sum_j a_{i,j} x^j$ and $Q_i(x) = \sum_q \binom{i+1}{q+1} x^q$ then $a_{i+1}(x) = a_i(x) \cdot Q_i(x)$.

Note, that above DP does not guarantees that selected inversion pairs occupy entire $\pi$. Let $b_{i,j}$ be as $a_{i,j}$ but additionally we guarantee that $M$ covers entire $\pi$. Then $b$ can be calculated as follows:

$$b_{i,j} = a_{i,j} - \sum_{k_1=1}^{i-1} b_{i-k,j} f(i,k) \qquad \text{where} \quad f(i,k) := \binom{i}{k}^2 k!$$

i.e. we take pairs $(\pi, M)$ with non-covered elements and subtract bad ones with $k$ non-covered elements. Note, formulas does not depend on $j$, so let us solve the problem for fixed $j$ and omit this index. Last formula could be rewritten as

$$b_i = \sum_{k_1+\ldots+k_c \le i} (-1)^c a_{i-k_1-\ldots-k_c} f(i,k_1) f(i-k_1,k_2) \cdot \ldots \cdot f(i-k_1-\ldots-k_{c-1},k_c)$$

where term

$$f(i,k_1) f(i-k_1,k_2) \cdot \ldots \cdot f(i-k_1-\ldots-k_{c-1},k_c) = \sum_{k_1+\ldots+k_c=i} (-1)^c \frac{i!^2}{k_1! \ldots k_c! (i-k_1-\ldots-k_{c-1})!^2}.$$

Let $a'_i = a_i / i!^2$ and $b'_i = b_i / i!^2$ and denote $c_i = \sum_{k_1+\ldots+k_c=i} (-1)^c \frac{1}{k_1! \ldots k_c!}$. Then again

$$b'_i = \sum_{x+y=i} a'_x \cdot c'_y$$

hence can be optimized with FFT as well. Finally we collect the answer using simple combinatorics. Recall that we need to calculate $n!$ once so we should precalc it up until modulo.

*Complexity:* $O(k^2 \log(k))$

# Problem Tutorial: "Mountains"

**Solution.** Consider such an array $a[1..n, 1..m]$ (that represents a valid map). Let's calculate a standard DP:

$$dp[i,j] = a[i,j] + \max(dp[i+1,j], dp[i,j+1]),$$

which stores the largest sum over all paths from $(i, j)$ to $(n, m)$. We can observe two facts:

- $dp[1,1] \le k$;
- $dp[i,j] \ge dp[i+1,j]$ and $dp[i,j] \ge dp[i+1,j]$ for $i = 1, \ldots, n$ and $j = 1, \ldots, m$.

Let's view matrix $dp$ as a pile of cubes: put $dp[i, j]$ cubes on top of the cell $(i, j)$. (see fig. 1). Such an object is called a *plane partition*. We will reduce the initial problem to the problem of counting plane partitions in a box $n \times m \times k$.

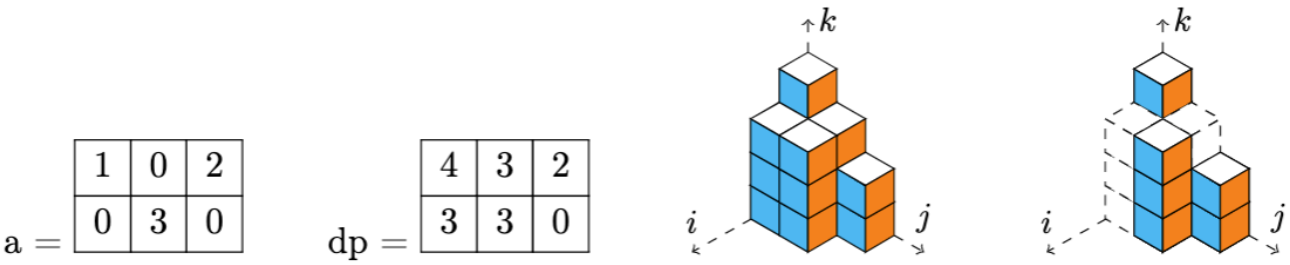



Figure 1. From left to right: array $a[i, j]$, $dp$ on array $a$, $dp$ as plane partition, 'corner flats' map to entries of $a$.

First, let's understand that the map $a \to dp$ is a bijection between $n \times m$ matrices with largest path $\le k$ and plane partitions in a box $n \times m \times k$. To show that, we can provide a simple inverse map: suppose that you are given a plane partition $dp[i, j]$, then

$$a[i, j] = dp[i, j] - \max(dp[i + 1, j], dp[i, j + 1]).$$

To understand the meaning of entries $a[i, j]$ in terms of a plane partition, take a look at the rightmost picture of Fig. 1.

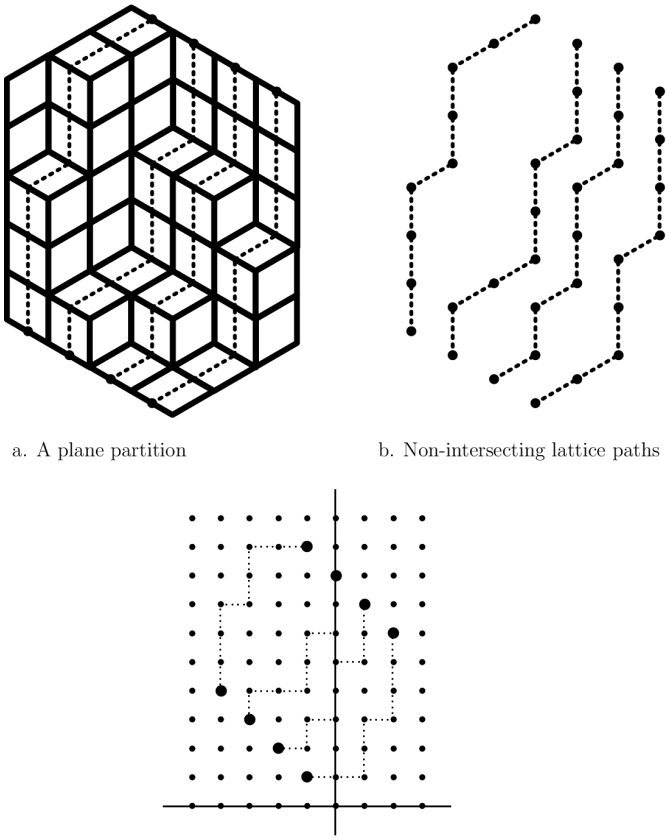
Now, to calculate the number of plane partitions in a box we can use several methods. One of the coolest is to view an arbitrary plane partition as a non-intersecting path system in a way represented in Fig. 2. Non-intersecting path systems in any planar directed acyclic graph can be enumerated by calculating the determinant of the 'path' matrix $f[i, j]$ = number of paths from $i \to j$ in the graph (by Lindstrom-Gessel-Viennot lemma). This can be done in $O(N^3)$ time with the Gaussian elimination algorithm.

Alternatively, entries $f_{i,j} = \binom{m+n}{n+j-i}$ are in fact binomial coefficients, so you can calculate the answer explicitly. This approach leads to a beautiful formula:

$$PP(n, m, k) = \prod_{i=1}^{n} \prod_{i=1}^{m} \prod_{i=1}^{k} \frac{i + j + k - 1}{i + j + k - 2},$$

which can be calculated in $O(N)$ time, but this was not required.

*Complexity:* $O(n^3)$



a. A plane partition

b. Non-intersecting lattice paths

Figure 2. Correspondence between plane partitions in a box and non-intersecting path systems.

*Open problem.* Calculate $PP(100, 100, 100, 100)$, i.e. number of DPs $dp[i, j, k] \le 100$ with similar inequalities in all directions.

# Problem Tutorial: "Kill All Termites"

We need to poison minimal number of vertices so that any path between two leaves contains poisoned vertex. Run DFS from the vertex of degree 2 and perform: if $v$ is leaf the set $a[v] = 1$, if $v$ is non-leaf the set $a[v] = \sum_{u \text{ - son of } v} a[u]$, and if $a[v] > 1$ then we poison this vertex and set $a[v] = 0$. This way we ensure that each path between leaves contains a poisoned vertex, on the other hand if some vertex $v$ encountered $a[v] > 1$ but were not poisoned, then there certainly is a poisonless path.

# Problem Tutorial: "Aidana and Pita"

Solution follows with "meet in the middle" approach. Let us split the array into two halves of size $\frac{n}{2}$ and for each part consider all possible $3^{n/2} < 1.6 \cdot 10^6$ distributions of dishes into 3 groups. Denote $L_1$ and $L_2$ as lists of resulting sums $(x, y, z)$ in left and right parts. Let $(x_1, y_1, z_1)$ be from $L_1$ and $(x_2, y_2, z_2)$ be from $L_2$. Since all 3! permutations of same distribution is in $L_2$ we may consider only the case when $x_1 + x_2 \ge y_1 + y_2 \ge z_1 + z_2$, then we want to minimize $x_1 + x_2 - z_1 - z_2 \to \min$.

Transform each triplet in $L_1$ by the rule $(x_1, y_1, z_1) \to (x_1 - y_1, y_1 - z_1)$ and each triplet from $L_2$ as $(x_2, y_2, z_2) \to (y_2 - x_2, z_2 - y_2)$. Then it is not hard to see, that for pair of points $(a_1, b_1) \in L_1$ and $(a_2, b_2) \in L_2$ we want to minimize $a_1 + b_1 - a_2 - b_2 \to \min$ having $a_1 \ge a_2$ and $b_1 \ge b_2$.

Merge lists $L_1$ and $L_2$ into $L$ and sort $L$ in increasing order of first coordinate. We will iterate over $L$ and for fixed $(a_1, b_1) \in L_1$ find the best fit. For that we maintain set $S$ of pairs $(b, a + b)$ of points from $L_2$ with both coordinates increasing. So we iterate over pairs $(a, b) \in L$ and think:

- if $(a, b)$ is from $L_1$ then take from $S$ point with maximal second coordinate (with max $a + b$),

- if $(a, b)$ is from $L_2$ then remove all the points $(b', a' + b') \in S$ with $b < b'$ but $a + b > a' + b'$, and the insert $(b, a + b) \to S$.

Then at each moment of time set $S$ will contain points of $L_2$ in weakly increasing order of both coordinates, since we move in increasing order of $a$ and remove non optimal solutions when needed.

*Time complexity:* $O(3^{n/2}n)$

# Problem Tutorial: "Box Packing"

Let us sort pairs $(a_i, b_i)$ lexicographically (first by $a_i$, if equal then by $b_i$). Then the problem is reduced to largest number of elements in $k$ disjoint weakly increasing subsequences of $b$. To solve this we use RSK algorithm.

First, recall the classical Longest Increasing Sequence search algorithm, that stores $d[i]$ = smallest value of last element of increasing subsequence of length $i$. The algorithm is following:

1. initially we have vector $d$ empty;
2. We insert number one by one, to insert $x$ we find smallest $j$ such that $x < d[j] = y$, then $x$ 'bumps' $y$, i.e. we set $d[j] = x$. Otherwise, if $x$ is not smaller than any number in the row we append it to the end. As a result, We have vector $d$ weakly increasing, so we can find suitable $j$ with bin-search.
3. after all numbers inserted, the size of $d$ is the answer.

Basically, RSK is an extension of this algorithm. We now have not a single row $d$, but several rows $d_1, d_2, \ldots$. Initially, they are all empty. Analogically, we insert number one by one as is (2), but instead of forgetting about the bumped value $y$ we insert it into the next row. This value will be either appended to the end of the current row or bump yet another number which is again inserted to the next row and so on.

As a result, we get sequence of rows $d_1, \ldots, d_m$ so that entries weakly increase in each row and strictly increase in each column (why?). Let $\lambda_i = size(d_i)$, then $\lambda$ is called partition of $n$ (sum of lengths is equal to $n$), a.k.a Young diagram if drawn as $\lambda_i$ empty boxes at row $i$. Let $\lambda'$ be a partition of the same diagram but by columns (for example $\lambda = 322$ then $\lambda' = 331$). It turns out to be a fact that:

**Theorem.**[Greene 1974] Let $\lambda$ be a diagram that results after applying RSK algorithm to the word $b_1, \ldots, b_n$. Then

- $\lambda_1 + \ldots + \lambda_k$ is equal to largest number of entries in the union of $k$ weakly increasing subsequences;
- $\lambda'_1 + \ldots + \lambda'_k$ is equal to largest number of entries in the union of $k$ strictly decreasing subsequences.

Using this, we need to store only first $k$ rows and ignore numbers going to the $k + 1$-st row.

*Complexity:* $O(nk \log(n))$

See the proof in *Stanley, Enumerative Combinatorics Vol. 2, Appendix A*. It is non-trivial, it would be interesting to find more direct argument.

*Proof sketch.* Let $w$ be a word and $RSK(w) = P$ of shape $\lambda$ and $I_k(w)$ is answer for initial problem. Show that the statement is true for permutation (or word) $w_0$ that is formed by reading tableau from bottom to top appending row by row (called reading word). Clearly $RSK(w_0) = P$. We show first that $I_k(w_0) = \lambda_1 + \ldots + \lambda_k$. Further we show, that $RSK(w) = RSK(w_0)$ if and only if $w \sim w_0$ where $\sim$ is so called Knuth equivalence relation on words. Finally we prove that $w \sim w_0$ and $I_k$ are preserved by Knuth transformations.

*Remark.* Telegram conversation yield an interesting question: how to restore the answer? One approach could be the following. We first find the answer for $w_0$ ($i$-th row of $RSK(w_0)$ is $i$-th subsequence). Further we find the sequence of Knuth moves from $w_0 \to w$ and change answer according to Knuth moves.

# Problem Tutorial: "Two Permutations"

**Problem.** Calculate the number of permutations $p, q$ of size n so that $\sum_{i=1}^{n} \max(p_i, q_i) = k$.

**Solution.** First of all, let's express our permutations as a table with two rows and fill it in increasing order. For example matrix after placing first 3 numbers:

| | | | | |
|---|---|---|---|---|
| 1 | 2 | | | 3 |
| 3 | 1 | | 2 | |

For this matrix, let $x$ be the number of complete columns, $y$ be the number of columns where only the upper cell is filled, and $z$ be the number of columns where only the lower cell is filled. The fact that we fill cells in increasing order of numbers implies that the the last placed number is immediately the greatest in the column and we can add it to the sum. Secondly, we can notice that whenever we placed the numbers from 1 to some $i$, $x$ is equal to $y$.

Let $dp[i][x][s]$ be the number of ways to place the numbers from 1 to $i$ in the table so that there are $x$ complete columns and the current sum is $s$, then

$$y = z = i - x$$

and we perform transitions as follows:

- we can put both of the $i$ in the empty columns:

$$dp[i+1][x][s] += dp[i][x][s];$$

- we can create one complete column and there are $y + z$ ways to do that:

$$dp[i+1][x+1][s+i] += dp[i][x][s] \cdot 2(i-x);$$

- We can create two complete columns and there are $x^2$ ways to do that:

$$dp[i+1][x+2][s+2i] += dp[i][x][s] \cdot x^2.$$

The last step is to optimize the memory usage. Let's notice that to calculate any $dp[i+1][x][s]$ you don't need to know $dp[i-1][x'][s']$. So at any given moment you just need to store two $n \times k$ matrices.

*Complexity:* $O(n^2 k)$

# Problem Tutorial: "Fancy Arrays"

Let's first consider a suboptimal solution. We write down all divisors of $m$ as $d_1 = 1, \ldots, d_k = m$ and create a matrix $D[i, j] = [\gcd(d_i, d_j) > 1]$. Then the answer is equal to the sum of entries of vector $(0, \ldots, 0, 1) \cdot D^n$. It can be computed with fast matrix exponentiation.

We now need to reduce the dimensions of that matrix. Let's think of what we really care about in transitions from one divisor to another:

- We can treat all primes with equal occurrences $\alpha_i = \alpha_j$ as indistinguishable. Let us group them together in $C_1, \ldots, C_g$, and for each group remember the corresponding occurrence $\alpha_i$;
- For each prime we only care if it is present in adjacent divisors, so we may think $\alpha_i = 0$ or 1 and store the number of non-zero powers. So for each group $C_i$ we can store the number of 1s in group $C_i$ satisfying $0 \le b_i \le |C_i|$.

After these optimizations, we will have $N \leq 255$ states, each of which can be described as $(b_1, \ldots, b_g)$ where $b_i$ denotes the number of primes from group $C_i$ with non-zero occurrences. Let $i$ and $j$ be the states described above. Note that for a fixed divisor $d_1$ of state $i$, the number of possible divisors $d_2$ of $j$ is the same for all $d_1$ from state $i$. Hence, we set $D[i, j]$ to be the number of transitions between states $i$ and $j$, which can be calculated combinatorially by "ALL - BAD" principle, and do matrix exponentiation of an $N \times N$ matrix $D$.

Finally, to answer the queries we can precalculate all powers of matrix $D$ that are powers of 2 $(D^1, D^2, D^4, D^8, \ldots)$ and answer each query in $N^3 \log(n)$ time.

*Complexity:* $O(qN^3 \log(n))$

# Problem Tutorial: "Restricted Arrays"

Let $G$ be graph on elements of an array. For each rule $(x, y)$ let us draw a directed edge $x \to y$ with weight 1 and edge $y \to x$ of weight $+1$. If we do unorientation of the graph then the edge means $x$ and $y$ can be recovered from each other. Now let us start at some vertex $v_0$, without loss of generality $a[v_0] = 0$ (if we have some valid array $a$ for some $M$ the we can shift all elements by $-a[v_0]$ then that would be valid array as well). Then start DFS from $v_0$ and calculate $a[v]$ for each $v$ in connected component of $v_0$. This way you will satisfy rules that are edges from DFS tree. To satisfy edge $(v, u)$ outside of DFS tree you want $abs(a[v] - a[u] + w(v, u))\text{mod } M == 0$ where $w(v, u)$ is a weight of an edge $(u, v)$. Thus any valid $M$ must be divisor of

$$N = \gcd_{(v,u) \text{ - back edges of dfs}}(abs(a[v] - a[u] + w(v, u))$$

Thus we calculate $N$ for each connected component, take gcd of all $N$s, store it in $N_0$. Then the answer would be all divisors of $N_0$.

Time complexity: $O(n + q \cdot \log(n))$.