Problem Tutorial: "Avg"

If k^x is not divisible by n for any $x > 0$ (or, equivalently, if n has a prime divisor that k doesn't have), no sequence of steps exists. To prove that, consider an array $A = (0, 0, \ldots, 0, 0, 1)$: each element of this array must be equal $\frac{1}{n}$ in the end, but after x steps we can only obtain rational values whose denominators are divisors of k^x.

Otherwise, a valid sequence always exists, and we can construct it inductively. If $n = k$, take $b_i = i$. Otherwise, find any $d > 1$ that is a divisor of k and $\frac{n}{k}$ (for example, $d = \gcd(k, \frac{n}{k})$). Split n elements into groups of size d. For each $\frac{k}{d}$ consecutive groups, perform a step equalizing them. Now the elements in each group are equal. Finally, form d groups of size $\frac{n}{d}$, one element from each group, and solve the problem recursively for each group.

The case when n is not divisible by k is more interesting. It's not even obvious how to check if a sequence exists: for example, when $n = 8$ and $k = 6$, it seems that there is no solution. If you have any insights about this, please share!