Problem Tutorial: "Bin"

Let's denote the given function as $f_k(n)$, then $f_k(n) = \sum\limits_{i=1}^{\min(n-1, \lfloor \frac{k+n}{2} \rfloor)} f_k(i) \cdot f_k(n-i)$.

Consider the following problem: we are given n integers $a_0, a_1, \ldots, a_{n-1}$ one by one, and after we get each a_i, we need to respond with $c_i = \sum_{j=0}^{i} a_j a_{i-j}$. If we can solve this problem in $O(T(n))$, we can solve the original problem in $O(T(n) + nk)$. Let's concentrate on this problem then.

If we are given all values of a_i in advance, we can just convolve the sequence with itself using NTT in $O(n \log n)$, however, the main difficulty lies in finding the coefficients of the convolution on the fly.

Let $f(L, R)$ be a recursive function that solves the problem for all $i \in [L; R)$, assuming that we already have coefficients up to $i-1$-th, and also b_i for $i \in [L; R)$ has been calculated as $b_i = \sum_{j=i-L+1}^{L-1} a_j a_{i-j}$.

If $R - L = 1$, read a_L and respond with $c_L = b_L + 2a_0 a_L$ if $L > 0$, and with $c_0 = a_0^2$ otherwise.

If $R - L > 1$, find $M = \lceil \frac{L+R}{2} \rceil$ and call $f(L, M)$ first. Now we want to call $f(M, R)$, but we need to recalculate b before that. If $L > 0$, convolve $a_L, a_{L+1}, \ldots, a_{M-1}$ and $a_0, a_1, \ldots, a_{M-L-1}$ using NTT, then add corresponding coefficients of this convolution to b_i for $i \geq M$. Otherwise, if $L = 0$, just convolve $a_0, a_1, \ldots, a_{M-1}$ with itself and replace b with the result.

Time complexity of this solution is $O(n \log^2 n)$.