Problem Tutorial: "Exp"

This problem might be well-known in some countries, but how do other countries learn about such problems if nobody poses them?

Consider a polynomial $P(y) = p_0 + p_1 y + \ldots + p_k y^k$. If we find the coefficients q_i of $Q(y) = P^n(y)$, the answer is $\sum_{i=0}^{nk} q_i \cdot \min(i, x)$. Since the sum of p_i is 1, we can also rewrite this as $\sum_{i=0}^{x-1} q_i \cdot i + (1 - \sum_{i=0}^{x-1} q_i) \cdot x$. Hence, we just need to find the first x coefficients of $P^n(y)$.

The title of this problem, Exp, stands for expected, experience, and exponentiation.

Consider the derivative of $P^{n+1}(y)$ and find two different expressions for it:

- $(P^{n+1}(y))' = (P(y)P(y)\ldots P(y))' = (n+1)P^n(y)P'(y) = A$;
- $(P^{n+1}(y))' = (P^n(y)P(y))' = (P^n(y))'P(y) + P^n(y)P'(y) = B$.

Since $A = B$, we have $nP^n(y)P'(y) = (P^n(y))'P(y)$. Consider the coefficient of y^i in both parts of this equation:

- in the left part, it's $n(q_i p_1 + 2q_{i-1}p_2 + \ldots + kq_{i-k+1}p_k)$;
- in the right part, it's $(i+1)q_{i+1}p_0 + iq_i p_1 + \ldots + (i-k+1)q_{i-k+1}p_k$.

It turns out that we can derive q_{i+1} from the equality of these two expressions if we know $q_0, q_1, \ldots, q_i$. Each coefficient can be calculated in $O(k)$, hence time complexity is $O(xk)$.