Problem Tutorial: "Grp"

The lower bound on the number of groups if k is odd is $\sum_{i=\frac{k+1}{2}}^{k} \binom{n}{i}$: all subsets of size at least $\frac{k+1}{2}$ have to belong to different groups. Similarly, if k is even, the lower bound is $\frac{1}{2}\binom{n}{k/2} + \sum_{i=\frac{k}{2}+1}^{k} \binom{n}{i}$.

Note that $\binom{n}{i} \geq \binom{n}{k-i}$ when $i \geq \frac{k}{2}$. Hence, if we can match all subsets of size i with subsets of size $k-i$ into non-intersecting pairs without common elements, we can achieve the lower bound.

Such a matching always exists when $i \neq k-i$, since the graph is bipartite and "regular" (not exactly, but all vertices in each part have equal degrees). When k is even, the graph is not bipartite, but it turns out that forming $\left\lfloor \frac{1}{2}\binom{n}{k/2} \right\rfloor$ pairs of subsets of size $\frac{k}{2}$ is always possible for $n \leq 17$. Even though the graphs are huge, we can build them and try to find maximum matchings: using Kuhn's algorithm for bipartite graphs, and using Edmonds' blossom algorithm (or maybe Kuhn's algorithm with hacks...) for non-bipartite graphs. Even though time complexity looks big, a greedy initialization already builds a huge part of the matching, and augmenting chains are very short on average too. You can try all possible test cases to make sure your solution is fast enough.

If you know a constructive way to build the matchings, or if you have a proof that an optimal matching of subsets of size $\frac{k}{2}$ for even k always exists, please share!