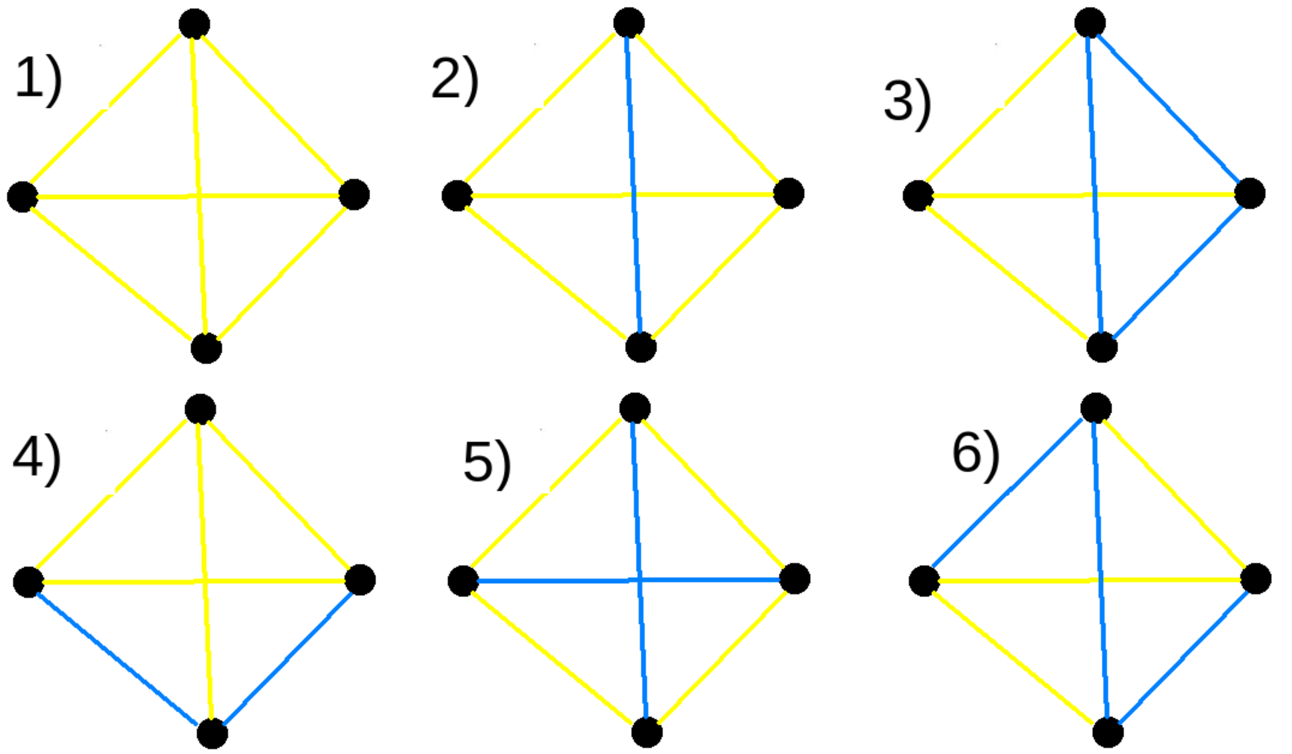
Problem Tutorial: "Glory Graph"

There are 6 non-isomoprhic types of graphs on 4 vertices (here we put complete blue and complete yellow graphs into the same type). Here they are. Let's denote the numbers of times they appear as subgraphs of G as $x_1, x_2, x_3, x_4, x_5, x_6$ correspondently.



We will now try to find some conditions on these numbers which would help us to dedude $Y - A = x_6 - x_2$. From now on, we will do some double counting.

- There are $\frac{n(n-1)(n-2)(n-3)}{24}$ subgraphs of size 4 in total. So $x_1+x_2+x_3+x_4+x_5+x_6 = \frac{n(n-1)(n-2)(n-3)}{24} = C_1$.
- Let's now double count the number of quadruples (A, B, C, D) of pairwise distinct vertices, where the edges AB and BC have different colors. From one side, it's equal to $0 \cdot x_1 + 8 \cdot x_2 + 12 \cdot x_3 + 12 \cdot x_4 + 16 \cdot x_5 + 16 \cdot x_6$. From other side, it's equal to $(n-3)$ by the sum of $2b_iy_i$ over all i, where b_i and y_i are the numbers of blue and yellow edges incident to vertex i respectively, and this sum can be calculated in $O(n^2)$. So, we can find $2x_2 + 3x_3 + 3x_4 + 4x_5 + 4x_6 = C_2$ in $O(n^2)$.
- Let's now double count the number of quadruples (A, B, C, D) of pairwise distinct vertices, where edges AB, BC, CD, DA, AC all have the same color. From one side, it's equal to $24 \cdot x_1 + 4 \cdot x_2 + 0 \cdot x_3 + 0 \cdot x_4 + 0 \cdot x_5 + 0 \cdot x_6$. From other side, it's equal to sum of $2cnt^1_{ij}(cnt^1_{ij} - 1)$ over all edges (i, j), where cnt^1_{ij} is the number of vertices k such that edges $(i, k), (j, k), (i, j)$ have the same color. We can calculate all cnt^1_{ij} in $O(\frac{n^3}{32})$ with bitsets. So, we can find $6x_1 + x_2 = C_3$ in $O(\frac{n^3}{32})$.
- Let's now double count the number of quadruples (A, B, C, D) of pairwise distinct vertices, where edges AB, BC, CD, DA all have the same color, which is different from the color of edge AC. From one side, it's equal to $0 \cdot x_1 + 4 \cdot x_2 + 0 \cdot x_3 + 0 \cdot x_4 + 8 \cdot x_5 + 0 \cdot x_6$. From other side, it's equal to sum of $2cnt^2_{ij}(cnt^2_{ij} - 1)$ over all edges (i, j), where cnt^2_{ij} is the number of vertices k such that edges $(i, k), (j, k)$ have the same color different from the color of edge (i, j). We can calculate all cnt^2_{ij} in $O(\frac{n^3}{32})$ with bitsets. So, we can find $x_2 + 2x_5 = C_4$ in $O(\frac{n^3}{32})$.

Now, we can write:

$$-3C_1 + C_2 + \frac{C_3}{2} - \frac{C_4}{2} =$$

$$= -3(x_1 + x_2 + x_3 + x_4 + x_5 + x_6) + (2x_2 + 3x_3 + 3x_4 + 4x_5 + 4x_6) + \frac{6x_1 + x_2}{2} - \frac{x_2 + 2x_5}{2} =$$
$$= x_6 - x_2$$

Note. While this may seem like a black magic, obtaining of this equation isn't that random: just write all the formulas you can with $x_1, x_2, \ldots, x_6$, and combine them to get $x_6 - x_2$, as the problem asks for that.