JOIRIS (Solution)

Let N be the number of columns, and K the length of pieces. The remainder of division of N by K is denoted by $N \bmod K$. In the following, the columns are numbered in 0-indexed, i.e. the columns are numbered from 0 to $N-1$. Let A_i be the number of blocks in the i-th column in the initial state of the game. Let B_i $(0 \le i \le K-1,\ 0 \le B_i \le K-1)$ be the sum of A_j's modulo K for j with $0 \le j \le N-1$ and $j \equiv i \pmod K$.

We claim that we can remove all blocks from the board if and only if both of the following conditions are satisfied:

- $B_0 = B_1 = \cdots = B_{(N \bmod K)-1}$
- $B_{N \bmod K} = B_{(N \bmod K)+1} = \cdots = B_{K-1}$

Here is a proof: by each of three possible operations (i.e. putting a piece horizontally/vertically, or deleting a row filled with blocks), the relative relations between $B_0, B_1, \ldots, B_{(N \bmod K)-1}$ and $B_{N \bmod K}, B_{(N \bmod K)+1}, \ldots, B_{K-1}$ do not change. (In other words, the equalities $B_0 = B_1 = \cdots = B_{(N \bmod K)-1}$ are satisfied after the operation if and only they are satisfied before the operation. The same is true for the equalities $B_{N \bmod K} = B_{(N \bmod K)+1} = \cdots = B_{K-1}$.) So, if at least one of the above equalities is not satisfied, we cannot remove all blocks from the board.

On the other hand, we can construct an algorithm to remove all blocks from the board in the following way.

Step 1 For $i = 1, \ldots, N-1$, if $A_{i-1} > A_i$, put pieces vertically on the i-th column unless $A_{i-1} \le A_i$. After this step, the board satisfies $A_0 \le A_1 \le \cdots \le A_{N-1}$, where A_i is the number of blocks in the i-th column.

Step 2 For $i = 0, \ldots, A_{N-1}$, put pieces horizontally on the i-th row in a right-justified way as many as possible. After this step, the board satisfies $A_{K-1} = A_K = \cdots = A_{N-1}$, i.e. the heights from the $(K-1)$-st column to the $(N-1)$-st column are the same.

Step 3 For $i = 0, \ldots, K-2$, put pieces vertically on the i-th column enough. After this step, the board satisfies $A_{K-1} = A_K = \cdots = A_{N-1} = 0$, i.e. all blocks from the $(K-1)$-st column to the $(N-1)$-st column disappear.

Step 4 Since $B_0 = B_1 = \cdots = B_{(N \bmod K)-1}$ and $B_{N \bmod K} = B_{(N \bmod K)+1} = \cdots = B_{K-1}$ in the initial state of the game, we have $A_0 \equiv A_1 \equiv \ldots \equiv A_{(N \bmod K)-1} \pmod K$ and $A_{N \bmod K} \equiv A_{(N \bmod K)+1} \equiv \cdots \equiv A_{K-1} \pmod K$ after Step 3. Since $A_{K-1} = 0$ after Step 3, we must have $A_{N \bmod K} \equiv A_{(N \bmod K)+1} \equiv \cdots \equiv A_{K-1} \equiv 0 \pmod K$. Putting pieces vertically, we can easily remove all blocks from the $(N \bmod K)$-th column to the $(K-2)$-nd column.

Step 5 After Step 4, the board satisfies $A_0 \equiv A_1 \equiv \ldots \equiv A_{(N \bmod K)-1} \pmod K$ and $A_{N \bmod K} = \cdots = A_{N-1} = 0$. Now we can easily remove all blocks from the board by putting pieces vertically from the 1-st column to the $((N \bmod K)-1)$-st column, and putting pieces horizontally from the $(N \bmod K)$-th column to the $(N-1)$-st column.

The proof of the claim is complete, and an algorithm is given.